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RESEARCH ARTICLE

A class of quasicontractive semigroups acting on Hardy and weighted Hardy spaces

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Abstract This paper studies semigroups of operators on Hardy and Dirichlet spaces whose generators are differential operators of order greater than one. The theory of forms is used to provide conditions for the generation of semigroups by second order differential operators. Finally, a class of more general weighted Hardy spaces is considered and necessary and sufficient conditions are given for an operator of the form $f \mapsto Gf^{(n_0)}$ (for holomorphic G and arbitrary n_0) to generate a semigroup of quasicontractions.

Keywords Strongly continuous semigroup · Hardy space · Weighted Hardy space · Quasicontractive semigroup · Form · Heat kernel

1 Introduction

The theory of semigroups of weighted composition operators on Hardy and Dirichlet spaces of the unit disc $\mathbb{D}$ is well understood (see [1,3–6,12,15,16]), and these have generators that are first order differential operators of the form $A : f \mapsto G_0 f + G_1 f'$.

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Less attention has been given to higher order generators (some of which are the counterpart of those appearing in the theory of heat semigroups), and that is the main theme of this paper.

In Sect. 2 we use the theory of forms to give sufficient conditions for the existence of semigroups on the Hardy space H^2 with generators given by second-order differential operators; other techniques involved in this section include perturbation theory and the use of generalized heat kernels in general reproducing kernel Hilbert spaces.

Then in Sect. 3 we consider a particular class of weighted Hardy spaces, including the Dirichlet space, and by means of numerical range techniques and the Lumer–Phillips theorem, together with explicit expressions for the reproducing kernels, we are able to provide necessary and sufficient conditions for generators of the form $f \mapsto Gf^{(n_0)}$ to generate a semigroup of quasicontractions.

Take $(\beta_n)_{n\geq 0}$ a sequence of positive real numbers. Then $H^2(\beta)$ is the space of analytic functions

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

in the unit disc that have finite norm

$$\|f\|_\beta = \left(\sum_{n=0}^{\infty} |c_n|^2 \beta_n^2 \right)^{1/2}.$$

The case $\beta_n = 1$ gives the usual Hardy space H^2; also, the case $\beta_0 = 1$ and $\beta_n = \sqrt{n}$ for $n \geq 1$ provides the Dirichlet space $\mathcal{D}$, which is included in $H^2(\mathbb{D})$.

With an extra condition on $(\beta_n)_n$, the Hilbert space $H^2(\beta)$ is also a *reproducing kernel space*, i.e., for all $w \in \mathbb{D}$, there exists a function $k_w \in H^2(\beta)$ such that

$$\langle f, k_w \rangle = f(w),$$

for all $f \in H^2(\beta)$ (see p. 19 in [7] and p. 146 in [13]). More precisely, if $(\beta_n)_n$ is such that

$$\sum_{n\geq 0} \frac{|w|^{2n}}{\beta_n^2} < \infty \quad \text{for all} \quad w \in \mathbb{D}, \tag{1}$$

it follows that $H^2(\beta)$ is a reproducing kernel Hilbert space and

$$k_w(z) = \sum_{n\geq 0} \frac{\overline{w}^n}{\beta_n^2} z^n \qquad \text{with} \qquad \|k_w\|^2_{H^2(\beta)} = \sum_{n\geq 0} \frac{|w|^{2n}}{\beta_n^2}.$$

In fact (1) is also equivalent to the more explicit condition $\liminf (\beta_n)^{1/n} \geq 1$.

We recall that H^∞ is the Banach algebra of bounded analytic functions in the disc, with the supremum norm.

Before embarking on the general theory we give a simple motivating example.

Example 1.1 Let $(Af)(z) = -z^2 f''(z)$ for $f \in D(A) \subset H^2$, where $D(A) = \{f \in H^2 : f'' \in H^2\}$. Note that if $f(z) = z^n$, then $(Af)(z) = -n(n-1)z^n$. It follows easily that A is the generator of the semigroup $(T(t))_{t \geq 0}$, where

$$T(t) \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n e^{-n(n-1)t} z^n.$$

2 Holomorphic semigroups on H^2

2.1 Application of the theory of forms

First we recall the definition of a form and we present the result that we will apply in order to obtain sufficient conditions for the existence of an analytic semigroup with a prescribed generator involving the second derivative.

The following presentation is based on [2], where we add a simplification concerning the mapping j defined below.

Let V be a complex Hilbert space and $a : V \times V \to \mathbb{C}$ a form, that is a continuous, sesquilinear and *coercive* mapping, i.e. there exists $\alpha > 0$ such that, for all $u \in V$,

$$\operatorname{Re} a(u, u) \geq \alpha \|u\|_V^2. \tag{2}$$

Then we take H another complex Hilbert space such that there is an embedding $j : V \to H$ continuous and dense range. Often we identify $j(v)$ with v for v in V.

For $x, y \in H$, we say that $x \in D(A)$ and $Ax = y$ if $x \in V$ and

$$a(x, v) = \langle y, v \rangle_H,$$

for all $v \in V$.

Then the linear operator A is well-defined, linear, densely-defined, with $A : D(A) \to H$.

Moreover, we say that a is j-elliptic if there exists $w \in \mathbb{R}$, $\alpha > 0$ such that

$$\operatorname{Re}(a(u, u)) + w\|u\|_H^2 \geq \alpha \|u\|_V^2. \tag{3}$$

The link between a and A is the following.

Theorem 2.1 [2, Thm.4.3] *If a is a form which is j-elliptic, then there exists a sector Σ_θ with $\theta \in (0, \pi/2]$ such that $-A$ generates a holomorphic C_0-semigroup of quasicontractions on H with*

$$\|T(t)\| \leq e^{\operatorname{Re}(t) w},$$

for all t in Σ_θ.

We now present an application of the above theory from forms to semigroups.

Take $H = H^2$, G_1, G_2, G_3 in H^∞ such that

$$\mathrm{Re}(G_1) \geq \varepsilon_1 \tag{4}$$

where ε_1 is a positive numerical constant.

Then define a by

$$a(f, g) = \langle G_1 f', g' \rangle_{H^2} + \langle G_2 f, g \rangle_{H^2} + \langle G_3 f', g \rangle_{H^2}$$

on $V \times V$, where $V := \{f \in H^2 : f' \in H^2\}$ is a Hilbert space endowed with the norm

$$\|f\|_V = \sqrt{\|f\|_{H^2}^2 + \|f'\|_{H^2}^2}.$$

Using the Cauchy–Schwarz inequality, a is continuous and obviously sesquilinear. It remains to check that a is j-elliptic.

Note that

$$\begin{aligned} \mathrm{Re}\,(a\,(f, f)) = & \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\left(G_1\left(e^{i\theta}\right)\right) |f'\left(e^{i\theta}\right)|^2 d\theta \\ & + \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\left(G_2\left(e^{i\theta}\right)\right) |f\left(e^{i\theta}\right)|^2 d\theta + \mathrm{Re}\left(\langle G_3 f', f \rangle_{H^2}\right) \end{aligned}$$

Moreover, once more using the Cauchy–Schwarz inequality, we have

$$\left|\mathrm{Re}\left(\langle G_3 f', f \rangle_{H^2}\right)\right| \leq \|G_3\|_\infty \|f'\|_{H^2} \|f\|_{H^2} \leq \frac{\|G_3\|_\infty}{2} \left(\|f'\|_{H^2}^2 + \|f\|_{H^2}^2\right).$$

Now for $\|G_3\|_\infty < 2\epsilon_1$, and for w sufficiently large, it follows that a is indeed j-elliptic.

The last step consists in identifying A. In our case A is defined by $Af = g$ when

$$\langle G_1 f', h' \rangle_{H^2} + \langle G_2 f, h \rangle_{H^2} + \langle G_3 f', h \rangle_{H^2} = \langle g, h \rangle_{H^2},$$

for all $h \in V$.

Note that for all $k, h \in V$, we have

$$\langle z(zk)', h \rangle_{H^2} = \langle k, h' \rangle_{H^2}.$$

Indeed, one can easily check this identity for k and h equal to powers of z.

It follows that A is defined on $D(A) \subset V$ by

$$Af = z\left(zG_1 f'\right)' + G_2 f + G_3 f'. \tag{5}$$

We have proved the following theorem which, as the referee has observed, is formally related to the Black–Scholes equation of mathematical finance (see [8,9]).

Theorem 2.2 *Let $G_1, G_2, G_3 \in H^\infty$ and $\epsilon_1 > 0$ be such that*

$$\mathrm{Re}(G_1) \geq \epsilon_1 \text{ and } \|G_3\|_\infty < 2\epsilon_1.$$

Let $A : D(A) \to H^2$ be defined by

$$Af(z) = z(zG_1(z)f'(z))' + G_2(z)f(z) + G_3(z)f'(z).$$

Then $-A$ generates a holomorphic C_0-semigroup of quasicontractions.

An easy example here is given by $G_1(z) = 1$, $G_2(z) = 0$ and $G_3(z) = -z$, in which case $Af(z) = z^2 f''(z)$, an example discussed in the introduction.

The proof of Theorem 2.2 leads to the following particular cases.

Theorem 2.3 *Let $G_1, G_2, G_3 \in H^\infty$ and $\epsilon_1 > 0$ be such that*

$$\mathrm{Re}(G_1) \geq \epsilon_1 \text{ and } \|G_3\|_\infty < 2\epsilon_1.$$

If moreover there exists $\varepsilon_2 > 0$ such that

$$\mathrm{Re}(G_2) \geq \epsilon_2 \text{ and } \|G_3\|_\infty < 2\epsilon_2,$$

then $-A$ defined by (5) generates a holomorphic C_0-semigroup of contractions.

Example 2.4 A particular case of Theorem 2.2 is the case where $Af(z) = z(zG_1(z)f'(z))'$ for which we can conclude that $-A$ generates a holomorphic C_0-semigroup of quasicontractions $(T(z))_{z\in\Sigma_\theta}$ with $\|T(z)\| \leq e^{w\mathrm{Re}(z)}$ for all $w > 0$. This implies that T is a holomorphic C_0-semigroup of contractions.

2.2 Perturbation theory

Recall that an operator B is bounded relative to another operator A if $D(A) \subseteq D(B)$ and there are constants $a, b > 0$ such that

$$\|Bx\| \leq a\|Ax\| + b\|x\| \qquad \text{for all} \quad x \in D(A). \tag{6}$$

We write $a_0 = \inf\{a > 0 : \exists b \text{ such that (6) holds}\}$, and call this the A-bound of B.

Theorem 2.5 [11, Thm. III.2.10] *Suppose that A generates an analytic semigroup $((T(z))_{z\in\Sigma_\theta\cup\{0\}}$. Then there is an $\alpha > 0$ such that $A + B$ (with $D(A + B) = D(A)$) generates an analytic semigroup for all B with A-bound $a_0 < \alpha$.*

More precisely, there is a $C \geq 1$ such that $\|(\lambda I - A)^{-1}\| \leq \frac{C}{|\lambda|}$ for λ in some large sector $\Sigma_{\pi/2+\delta}$ with $\delta > 0$. Then we may take $\alpha = 1/(C+1)$.

To see an application of this result, let us take A defined by $Af = -z^2 f''$ for $f \in D(A) \subset H^2$, which is diagonalisable with $Ae_n = -n(n-1)e_n$, where $e_n(z) = z^n$ for $n = 0, 1, 2, \ldots$

Now

$$(\lambda I - A)^{-1} e_n = \frac{e_n}{\lambda + n(n-1)}.$$

Suppose that $\lambda = -x + iy$ with $|x/y| \leq \epsilon$ (the case $x \geq 0$ is easier); then

$$\frac{1}{\sqrt{(-x + n(n-1))^2 + y^2}} \leq \frac{1}{|y|} = \frac{\sqrt{|x/y|^2 + 1}}{\sqrt{x^2 + y^2}} \leq \frac{\sqrt{1+\epsilon^2}}{|\lambda|},$$

so we may take $C = \sqrt{1+\epsilon^2}$ for any $\epsilon > 0$, and hence $\alpha = \frac{1}{1+\sqrt{1+\epsilon^2}}$. Thus, taking $a_0 = 1/2$. we may apply Theorem 2.5, and conclude the following.

Theorem 2.6 *Let $Af = -z^2 f''$ and $Bf = gf''$, where $\|g\|_\infty \leq \frac{1}{2}$. Then $A + B$ generates a holomorphic semigroup on H^2.*

2.3 Heat kernels

It is well known (see, e.g., [10]) that many parabolic partial differential equations on $\mathbb{R}^n$ can be solved in terms of a (generalized) heat kernel. The simplest example is defined on $L^2(\mathbb{R})$ by

$$T(t)f(s) = (4\pi t)^{-1/2} \int_{\mathbb{R}} f(r) e^{-(s-r)^2/4t} \, dr,$$

giving a semigroup whose generator is the closure of the Laplacian $f \mapsto f''$.

Let H be a reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$. For a semigroup $(T(t))_{t \geq 0}$ on H with generator $f \mapsto \sum_{k=0}^{n_0} G_k f^{(k)}$, where the G_k are analytic in $\mathbb{D}$, define the associated kernel $K(t, z, w)$ on $\mathbb{R}_+ \times \mathbb{D} \times \mathbb{D}$ by

$$K(t, z, w) = T(t) k_w(z),$$

where k_w is the reproducing kernel for H. That is, K satisfies the equation

$$\frac{\partial K}{\partial t} = \sum_{k=0}^{n_0} G_k \frac{\partial^k K}{\partial z^k}, \qquad K(0, z, w) = k_w(z). \tag{7}$$

Proposition 2.7 *Suppose that $T(t)f(w) = F(t, w)$; then*

$$F(t, w) = \langle f, K(t, z, w) \rangle_z, \qquad (f \in H).$$

Proof This is clearly true if f is a finite linear combination of reproducing kernels, and it follows for all f by the boundedness of the operator $T(t)$. □

Theorem 2.8 *If the equation* $\sum_{k=0}^{n_0} G_n \varphi^{(k)} = \lambda\varphi$ *has a normalized basis* (φ_n) *of eigenvectors in* H*, with eigenvalues* (λ_n)*, such that* $\operatorname{Re}\lambda_n \le 0$ *for all* n*, then*

$$K(t,z,w) = \sum \varphi_n(z)\overline{\varphi_n(w)}e^{-\lambda_n t}.$$

Proof By a simple calculation $f(w) = \langle f, K(0,z,w)\rangle_z$ for $f \in H$, and so $K(0,z,w) = k_w(z)$. Moreover, the hypotheses on (φ_n) and λ_n easily imply that (7) holds. □

Example 2.9 For $H = H^2$, with reproducing kernel $k_w(z) = 1/(1-\overline{w}z)$, consider the generator $f \mapsto Gf''$, where $G(z) = -z^2$, and $\varphi_n(z) = z^n$, $\lambda_n = -n(n-1)$. Then

$$K(t,z,w) = \sum_{n=0}^{\infty} e^{-n(n-1)t}\overline{w}^n z^n.$$

Take $f(z) = \sum_{n=0}^{\infty} a_n z^n$; then $F(t,w) = T(t)f(w)$ is given by

$$F(t,w) = \sum_{n=0}^{\infty} a_n e^{-n(n-1)t} w^n.$$

3 Link between the generator and the numerical range

Let n_0 be a positive integer and let G be an analytic function in the disc (we impose a more general condition on G later).

Assume that A is defined on a dense set of H^2 by

$$Af = Gf^{(n_0)},$$

where $f^{(n_0)}$ denotes the derivative of f of order n_0.

If $G(z) = \sum_{n=0}^{\infty} a_n z^n$, denote by $\widetilde{G}$ the analytic function associated with G defined by

$$\widetilde{G}(z) = a_{n_0} + \sum_{n=1}^{\infty}\left(a_{n_0+n} + \overline{a_{n_0-n}}\right) z^n,$$

with the convention $a_j = 0$ if $j < 0$.

Note that, for all $z \in \mathbb{T}$,

$$\operatorname{Re}\left(\overline{z^{n_0}}G(z)\right) = \operatorname{Re}\left(\widetilde{G}(z)\right). \tag{8}$$

Consider now the weighted Hardy space $H^2(\beta)$ where β is the sequence defined by

$$\beta_n = \begin{cases} 1 & \text{for } 1 \leq n < n_0 \\ \sqrt{\frac{(n-1+n_0)!}{(n-1)!}} & \text{for } n \geq n_0. \end{cases}$$

In other words, for this particular weight β, we have $H^2(\beta) \subset H^2$ and

$$f \in H^2(\beta) \Longleftrightarrow \langle f, f \rangle_{H^2(\beta)} := \sum_{n=0}^{n_0-1} |a_n|^2 + \sum_{n=n_0}^{\infty} \frac{(n-1+n_0)!}{(n-1)!} |a_n|^2 < \infty.$$

Note also that there is a link between the scalar product $\langle , \rangle_{H^2(\beta)}$ and the usual scalar product on H^2 denoted by $\langle , \rangle_{H^2}$, namely, for all $f, g \in H^2(\beta)$, we have

$$\langle f, g \rangle_{H^2(\beta)} = \langle f, z^{n_0} g^{(n_0)} \rangle_{H^2} + \sum_{n=0}^{n_0-1} \frac{f^{(n)}(0)}{n!} \overline{\frac{g^{(n)}(0)}{n!}}.$$

We can now present a link between a condition on G and the upper boundedness of the numerical range of A acting on $H^2(\beta)$.

Proposition 3.1 *If* $\operatorname{ess\,sup}_{w \in \mathbb{T}} \operatorname{Re}(\overline{w^{n_0}} G(w)) \leq 0$*, and* $D(A)$ *is dense in* $H^2(\beta)$*, where* A *is defined by* $Af = Gf^{(n_0)}$ *then*

$$\sup \operatorname{Re} \left\{ \langle Af, f \rangle_{H^2(\beta)} : f \in D(A), \|f\|_{H^2(\beta)} = 1 \right\} < \infty.$$

Proof Writing T_F for the Toeplitz operator $f \mapsto P_{H^2} Ff$, we have

$$\begin{aligned}
\operatorname{Re}\langle Af, f \rangle_{H^2(\beta)} &= \operatorname{Re}\langle Gf^{(n_0)}, f \rangle_{H^2(\beta)} \\
&= \operatorname{Re}\langle Gf^{(n_0)}, z^{n_0} g^{(n_0)} \rangle_{H^2} + \operatorname{Re}\left(\sum_{n=0}^{n_0-1} \frac{(Gf^{(n_0)})^{(n)}(0)}{n!} \overline{\frac{f^{(n)}(0)}{n!}} \right) \\
&= \operatorname{Re}\langle T_{\overline{z^{n_0}} G} f^{(n_0)}, f^{(n_0)} \rangle_{H^2} + \operatorname{Re}\left(\sum_{n=0}^{n_0-1} \frac{(Gf^{(n_0)})^{(n)}(0)}{n!} \overline{\frac{f^{(n)}(0)}{n!}} \right) \\
&= \langle T_{\operatorname{Re}(\overline{z^{n_0}} G)} f^{(n_0)}, f^{(n_0)} \rangle_{H^2} + \operatorname{Re}\left(\sum_{n=0}^{n_0-1} \frac{(Gf^{(n_0)})^{(n)}(0)}{n!} \overline{\frac{f^{(n)}(0)}{n!}} \right) \\
&= \langle T_{\operatorname{Re}(\widetilde{G})} f^{(n_0)}, f^{(n_0)} \rangle_{H^2} + \operatorname{Re}\left(\sum_{n=0}^{n_0-1} \frac{(Gf^{(n_0)})^{(n)}(0)}{n!} \overline{\frac{f^{(n)}(0)}{n!}} \right) \\
&= \operatorname{Re}\langle \widetilde{G} f^{(n_0)}, f^{(n_0)} \rangle_{H^2} + \operatorname{Re}\left(\sum_{n=0}^{n_0-1} \frac{(Gf^{(n_0)})^{(n)}(0)}{n!} \overline{\frac{f^{(n)}(0)}{n!}} \right).
\end{aligned}$$

It follows that $\operatorname{ess\,sup}_{z\in\mathbb{T}} \operatorname{Re}(\widetilde{G}(z)) \leq 0$ implies that

$$\operatorname{Re}\langle Af, f\rangle_{H^2(\beta)} \leq \operatorname{Re}\left(\sum_{n=0}^{n_0-1} \frac{(Gf^{(n_0)})^{(n)}(0)}{n!} \frac{\overline{f^{(n)}(0)}}{n!}\right) \leq (n_0!)^2 \|Gf^{(n_0)}\|_{H^2}.$$

In order to find an upper bound independent of the choice of f, note that we may assume without loss of generality that G and f are polynomials of degree at most $2n_0$. Moreover since the norm of f in $H^2(\beta)$ is 1, in particular the Taylor coefficients of f are bounded by 1. It is now clear that there exists a numerical constant $C > 0$ depending only on n_0 and the norm of G in $H^2(\beta)$, such that

$$\operatorname{Re}\langle Af, f\rangle_{H^2(\beta)} \leq C,$$

for all f in the unit ball of $H^2(\beta)$. □

The following result, which serves as a converse, applies in a large family of weighted Hardy spaces with reproducing kernels. Now the sequence $(\beta_n)_n$ need not depend on the operator A.

Proposition 3.2 *Let $(\beta_n)_n$ be a decreasing sequence of positive reals such that $\liminf_{n\to\infty} |\beta_n|^{1/n} \geq 1$ and let $G \in H^2(\beta)$ such that*

$$\operatorname*{ess\,sup}_{w\in\mathbb{T}} \operatorname{Re}(\overline{w^{n_0}} G(w)) > 0.$$

Then

$$\sup \operatorname{Re}\left\{\langle Af, f\rangle_{H^2(\beta)} : f \in D(A), \|f\|_{H^2(\beta)} = 1\right\} = +\infty,$$

where A is defined on $D(A) = \{f \in H^2(\beta) : Gf^{(n_0)} \in H^2(\beta)\}$ by $Af = Gf^{(n_0)}$.

Before proceeding to the proof, we state the following technical lemma which explains the hypothesis on monotonicity of $(\beta_n)_n$.

Lemma 3.3 *Let $(\beta_n)_n$ be a decreasing sequence of positive reals. Then for each positive integer N, there exists $\eta = \eta(N) > 0$ such that for all $z \in \mathbb{D}$ with $|w| > 1-\delta$, we have*

$$\sum_{n=0}^{N} \frac{|w|^{2n}}{\beta_n^2} < \sum_{n=N+1}^{\infty} \frac{|w|^{2n}}{\beta_n^2}.$$

Proof Since $(1/\beta_n)_n$ is increasing, we have

$$\sum_{n=0}^{N} \frac{|w|^{2n}}{\beta_n^2} \leq \frac{1}{\beta_N^2}\left(1 + |w|^2 + \cdots + |w|^{2N}\right) = \frac{1}{\beta_N^2}\left(\frac{1-|w|^{2N+2}}{1-|w|^2}\right).$$

On the other hand, we have

$$\sum_{n=N+1}^{\infty}\frac{|w|^{2n}}{\beta_n^2}\geq\frac{1}{\beta_{N+1}^2}\sum_{n=N+1}^{\infty}|w|^{2n}=\frac{|w|^{2N+2}}{\beta_{N+1}^2(1-|w|^2)}\geq\frac{|w|^{2N+2}}{\beta_N^2(1-|w|^2)}.$$

Since $1-|w|^{2N+2}<|w|^{2N+2}$ is equivalent to $|w|>(1/2)^{1/(2N+2)}$, for all $w\in\mathbb{D}$ such that $|w|>\eta(N)$ with $\eta(N)=1-(1/2)^{1/(2N+2)}$, we have

$$\sum_{n=0}^{N}\frac{|w|^{2n}}{\beta_n^2}<\sum_{n=N+1}^{\infty}\frac{|w|^{2n}}{\beta_n^2}.$$

□

Proof of Proposition 3.2 By hypothesis, there exists $\delta>0$ and a sequence $(w_k)_k\subset\mathbb{D}$ such that $|w_k|\to 1$ and $\mathrm{Re}(\overline{w_k^{n_0}}G(w_k))\geq\delta$. Moreover the condition $\liminf_{n\to\infty}|\beta_n|^{1/n}\geq 1$ guarantees that the space $H^2(\beta)$ has reproducing kernels k_w for all $w\in\mathbb{D}$. Now consider the sequence $(\widehat{k_{w_k}})_k$ of normalized reproducing kernels associated with $(w_k)_k$, i.e. $\widehat{k_{w_k}}=\frac{k_{w_k}}{\|k_{w_k}\|_{H^2(\beta)}}$. First assume that $k_{w_k}\in D(A)$. In this case, the remainder of the proof consist in checking that

$$\lim_{k\to\infty}\mathrm{Re}\left(\langle A\widehat{k_{w_k}},\widehat{k_{w_k}}\rangle_{H^2(\beta)}\right)=+\infty.$$

Note that

$$\langle A\widehat{k_{w_k}},\widehat{k_{w_k}}\rangle_{H^2(\beta)}=\langle G\left(\widehat{k_{w_k}}\right)^{(n_0)},\widehat{k_{w_k}}\rangle_{H^2(\beta)}=\frac{1}{\|k_{w_k}\|^2_{H^2(\beta)}}G\left(w_k\right)k_{w_k}^{(n_0)}\left(w_k\right),$$

where $k_{w_k}^{(n_0)}(z)=\sum_{n\geq n_0}\frac{n(n-1)\cdots(n-n_0+1)\overline{w_k}^n}{\beta_n^2}z^{n-1}$.

It follows that

$$\langle Ak_{w_k},k_{w_k}\rangle_{H^2(\beta)}=\sum_{n\geq 1}\frac{n(n-1)\cdots(n-n_0+1))G(w_k)\overline{w_k^{n_0}}|w_k|^{2(n-n_0)}}{\beta_n^2},$$

and thus

$$\langle A\widehat{k_{w_k}},\widehat{k_{w_k}}\rangle_{H^2(\beta)}=\frac{\overline{w_k^{n_0}}G(w_k)}{|w_k|^{2n_0}}\frac{\sum_{n\geq n_0}\frac{n(n-1)\cdots(n-n_0+1)|w_k|^{2n}}{\beta_n^2}}{\sum_{n\geq 0}\frac{|w_k|^{2n}}{\beta_n^2}}.$$

Now, for each positive integer N, take $\eta(N)$ as in Lemma 3.3, and k sufficiently large so that $|w_k| > 1 - \eta(N)$. Then we have

$$\begin{aligned}
\frac{\sum_{n\geq n_0} \frac{n(n-1)\cdots(n-n_0+1))|w_k|^{2n}}{\beta_n^2}}{\sum_{n\geq 0} \frac{|w_k|^{2n}}{\beta_n^2}} &= \frac{\sum_{n=n_0}^{N} \frac{n(n-1)\cdots(n-n_0+1)|w_k|^{2n}}{\beta_n^2}}{\sum_{n\geq 0}^{N} \frac{|w_k|^{2n}}{\beta_n^2} + \sum_{n=N+1}^{\infty} \frac{|w_k|^{2n}}{\beta_n^2}} \\
&\quad + \frac{\sum_{n=N+1}^{\infty} \frac{n(n-1)\cdots(n-n_0+1)|w_k|^{2n}}{\beta_n^2}}{\sum_{n\geq 0}^{N} \frac{|w_k|^{2n}}{\beta_n^2} + \sum_{n=N+1}^{\infty} \frac{|w_k|^{2n}}{\beta_n^2}} \\
&\geq \frac{(N+1)N\cdots(N+1-n_0+1)\sum_{n=N+1}^{\infty} \frac{|w_k|^{2n}}{\beta_n^2}}{2\sum_{N+1}^{\infty} \frac{|w_k|^{2n}}{\beta_n^2}} \\
&= \frac{(N+1)N\cdots(N+1-n_0+1)}{2}.
\end{aligned}$$

Therefore, for k sufficiently large (so that $|w_k| > 1 - \eta(N)$), we get

$$\operatorname{Re}\left(\langle A\widehat{k_{w_k}}, \widehat{k_{w_k}}\rangle_{H^2(\beta)}\right) \geq \frac{(N+1)N\cdots(N+1-n_0+1)}{2|w_k|^{2n_0}} \operatorname{Re}(\overline{w_k^{n_0}}G(w_k)).$$

Since $\operatorname{Re}(\overline{w_k^{n_0}}G(w_k)) \geq \delta$ and since $|w_k|$ tends to 1, we get the desired conclusion.

If k_{w_k} is not in $D(A)$, the conclusion follows from similar calculation, considering the sequence of polynomials $(k_{w_k}^M)_{M\geq 0}$ defined by

$$k_{w_k}^M = \sum_{n=0}^{M} \frac{\overline{w_k^n}}{\beta_n^2} z^n,$$

which belongs to $D(A)$ and tends to k_{w_k} in $H^2(\beta)$. □

Corollary 3.4 *For A to generate a C_0 semigroup of quasicontractions on $H^2(\beta)$ it is necessary and sufficient that* $\operatorname{ess\,sup}_{w\in\mathbb{T}} \operatorname{Re}(\overline{w^{n_0}}G(w)) \leq 0$ *and there is a $\lambda > 0$ such that $A - \lambda I$ is invertible in $H^2(\beta)$. Moreover, such a semigroup is holomorphic if and only if there is a $\beta \in (0, \pi/2)$ such that* $\operatorname{ess\,sup}_{w\in\mathbb{T}} \operatorname{Re}(e^{\pm i\beta}\overline{w^{n_0}}G(w)) \leq 0$.

Proof The first part follows from Propositions 3.1 and 3.2 using the Lumer–Phillips theorem [14, p. 14]. The characterization of holomorphy follows from the complex version of the Lumer–Phillips theorem as in [5, Prop. 2.2]. □

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References

1. Abate, M.: The infinitesimal generators of semigroups of holomorphic maps. Ann. Mat. Pura Appl. **161**(4), 167–180 (1992)
2. Arendt, W., ter Elst, A.F.M.: From forms to semigroups. In: Spectral Theory, Mathematical System Theory, Evolution Equations, Differential and Difference Equations. Operator Theory: Advances and Applications, vol. 221, pp. 47–69. Birkhäuser/Springer Basel AG, Basel (2012)
3. Arvanitidis, A.G.: Semigroups of composition operators on Hardy spaces of the half-plane. Acta Sci. Math. (Szeged) **81**(1–2), 293–308 (2015)
4. Avicou, C., Chalendar, I., Partington, J.R.: A class of quasicontractive semigroups acting on Hardy and Dirichlet space. J. Evol. Equ. **15**(3), 647–665 (2015)
5. Avicou, C., Chalendar, I., Partington, J.R.: Analyticity and compactness of semigroups of composition operators. J. Math. Anal. Appl. **437**(1), 545–560 (2016)
6. Berkson, E., Porta, H.: Semigroups of analytic functions and composition operators. Mich. Math. J. **25**(1), 101–115 (1978)
7. Cowen, C.C., MacCluer, B.D.: Composition Operators on Spaces of Analytic Functions. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL (1995)
8. Emamirad, H., Goldstein, G.R., Goldstein, J.A.: Chaotic solution for the Black–Scholes equation. Proc. Am. Math. Soc. **140**(6), 2043–2052 (2012)
9. Emamirad, H., Goldstein, G.R., Goldstein, J.A.: Corrigendum and improvement to "Chaotic solution for the Black–Scholes equation". Proc. Am. Math. Soc. **142**(12), 4385–4386 (2014)
10. Engel, K.J., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations. Springer, New York (2000)
11. Engel, K.J., Nagel, R.: A Short Course on Operator Semigroups. Universitext. Springer, New York (2006)
12. König, W.: Semicocycles and weighted composition semigroups on H^p. Mich. Math. J. **37**(3), 469–476 (1990)
13. Partington, J.R.: Interpolation, Identification, and Sampling. London Mathematical Society Monographs. New Series, vol. 17. Oxford University Press, New York (1997)
14. Pazy, A.: Semigroups of linear operators and applications to partial differential equations. Applied Mathematical Sciences, vol. 44. Springer, Berlin (1983)
15. Siskakis, A.G.: Semigroups of composition operators on the Dirichlet space. Results Math. **30**, 165–173 (1996)
16. Siskakis, A.G.: Semigroups of composition operators on spaces of analytic functions, a review. Studies on Composition Operators (Laramie, WY, 1996). Contemp. Math., vol. 213, pp. 229–252. American Mathematical Society, Providence (1998)